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THE FLOW PATTERN IN A LIQUID LAYER AND THE SPECTRUM OF THE BOUNDARY-VALUE PROBLEM WHEN THE SURFACE TENSION DEPENDS NON-LINEARLY ON THE TEMPERATURE†

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The flow of a liquid in a plane channel on the bottom of which a specified temperature distribution is maintained while the free surface is thermally isolated is considered. The surface tension of the liquid depends quadratically on the temperature. The system of Navier–Stokes and heat conduction equations possess a self-similar solution which leads to the non-linear eigenvalue problem of finding the flow temperature fields in the channel. The spectrum of this problem is investigated analytically for low Marangoni numbers (the second approximation) and numerically in the limiting case of an ideally heat conducting liquid for any Marangoni number. The pattern of the thermocapillary flow in the layer is analysed as a function of the parameter values. The non-uniqueness of the solution, which is typical for problems of this kind, is established. The results are compared with those obtained previously in the first approximation with respect to the Marangoni number. © 1997 Elsevier Science Ltd. All rights reserved.

Capillary effects under conditions of low or compensated gravity constitute a class of phenomena which have attracted the attention of many investigators [1]. The growing of crystals and the creation of composites with new properties at low gravity and the preparation of ultrapure metals and glasses in space by the thermocapillary deposition of drops and bubbles of an extraneous phase represent a far from complete list of the applications of the effects considered. The temperature dependence of the surface tension is one of the important factors which determine the diversity of the dynamics of the interfacial surface when there is a non-uniform temperature field in a system.

The problem of the thermocapillary convection of a weightless liquid in a plane layer with a free thermally isolated surface which is heated from the bottom has previously been considered in [2] using the Navier–Stokes equations when the surface tension depends quadratically on the temperature. By separation of the variables, a two-point boundary-value problem was obtained which describes the motion of the liquid and the temperature distribution in the layer. The spectrum of this problem is investigated below.

1. FORMULATION OF THE PROBLEM. INITIAL EQUATIONS

Suppose that a viscous incompressible and weightless liquid forms a plane horizontal layer of thickness H on a solid non-uniformly heated surface with a temperature distribution $T = T_0 + AX$, where T_0 and A are constants and X is a horizontal coordinate. The origin of the Cartesian system of coordinates XOY (Y is the vertical coordinate measured from the base, transverse to the layer) is placed at a point on the base of the layer at a temperature T_0. The free surface of the layer is thermally isolated and the surface tension in it depends quadratically on the temperature as given by $\sigma(T) = \sigma_0 + 1/2\alpha(T - T_0)^2$, where σ_0 and α are constants. Values of $\alpha > 0$ ($\alpha < 0$) correspond to a parabolic dependence of $\sigma(T)$ with a local minimum (maximum). Steady flow in the layer, which corresponds to the balance between the tangential thermocapillary stresses and the viscous stresses on the free surface, is described by the system of equations and boundary conditions

$$(\mathbf{v}\nabla)\mathbf{v} = -\rho^{-1}\nabla p + \nu\nabla^2\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0$$
$$(\mathbf{v}\nabla)T = \chi\nabla^2 T$$
$$\mathbf{v} = (u, v), \quad \nabla = (\partial/\partial X, \partial/\partial Y) \tag{1.1}$$

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$$Y=0,\quad u=0,\quad v=0,\quad T=T_0+AX$$
$$Y=H,\quad v=0,\quad \partial T/\partial Y=0,\quad \nu\rho\partial u/\partial Y=\partial\sigma/\partial X=\alpha(T-T_0)\partial T/\partial X$$

Here ρ, p and $\mathbf{v}$ are the density, pressure and velocity (the components along the X and Y axes are u and v, respectively), ν is the kinematic viscosity and χ is the thermal diffusivity of the liquid.

Changing to dimensionless coordinates ($x = X/H$, $y = Y/H$) and introducing the dimensionless stream function $x\psi(y)$, temperature $x\theta(y)$ and pressure $\lambda x^2 + f(y)$, using the relations

$$u=\frac{\nu}{H}x\psi'(y),\quad v=-\frac{\nu}{H}\psi(y),\quad T=T_0+AHx\theta(y)$$
$$p=p_0-\frac{\rho}{2}\left(\frac{\nu}{H}\right)^2[\lambda x^2+f(y)]$$

where p_0 is the pressure at the origin of coordinates and λ is a pressure coefficient, system (1.1) reduces [2] to the non-linear two-point boundary-value problem

$$\psi'''+\psi\psi''-\psi'^2+\lambda=0 \tag{1.2}$$

$$\theta''-\Pr(\psi'\theta-\psi\theta')=0 \tag{1.3}$$

$$\begin{gathered}y=0,\quad \psi=0,\quad \psi'=0,\quad \theta=1\\ y=1,\quad \psi=0,\quad \psi''=m_H\theta^2,\quad \theta'=0\end{gathered} \tag{1.4}$$

for determining the unknown functions $\psi(y)$, $\theta(y)$ ($f(y) = \psi^2(y) + 2\psi'(y)$).

Here $\Pr = \nu/\chi$ and $m_H = \alpha A^2H^3/(\rho\nu^2)$ are the Prandtl and Marangoni numbers, which are the parameters of the problem, while the coefficient λ plays the part of an eigenvalue.

The boundary-value problem (1.2)–(1.4) has been investigated [2] by the method of small perturbations in the limiting case when $m_H \to 0$, $\lambda \to 0$, and the corresponding asymptotic form of the spectrum $\lambda = \lambda(m_H, \Pr)$ was obtained in the form

$$\lambda=-\tfrac{3}{2}m_H \tag{1.5}$$

where, in the first approximation with respect to the Marangoni number, neither the asymptotic form itself nor the velocity field in the layer depend on the Prandtl number.

2. THE SECOND APPROXIMATION IN THE MARANGONI NUMBER

The expansions of the functions ψ, θ, f and the eigenvalue λ in series in m_H, apart from terms $o(m_H^2)$, have the form

$$\psi=m_H\psi_1+m_H^2\psi_2,\quad \theta=1+m_H\theta_1+m_H^2\theta_2$$
$$f=m_Hf_1+m_H^2f_2,\quad \lambda=m_H\lambda_1+m_H^2\lambda_2$$

Here [2]

$$\begin{gathered}\psi_1=\frac{y^2}{4}(y-1),\quad \theta_1=-\frac{\Pr}{16}y^3\left(\frac{4}{3}-y\right)\\ f_1=y\left(\frac{3}{2}y-1\right),\quad \lambda_1=-\frac{3}{2}\end{gathered} \tag{2.1}$$

In the case of the second approximation, the boundary-value problem has the form

$$\psi_2''' + \psi_1\psi_1'' - \psi_1'^2 + \lambda_2 = 0, \quad f_2 = \psi_1^2 + 2\psi_2' \tag{2.2}$$

$$\theta_2'' - \Pr(\psi_1'\theta_1 + \psi_2' - \psi_1\theta_1') = 0 \tag{2.3}$$

$$y = 0: \quad \psi_2 = 0, \quad \psi_2' = 0, \quad \theta_2 = 0 \tag{2.4}$$

$$y = 1: \quad \psi_2 = 0, \quad \psi_2'' = 2\theta_1, \quad \theta_2' = 0 \tag{2.5}$$

Using (2.1), we obtain

$$\psi_2''' = \frac{3}{16}y^4 - \frac{y^3}{4} + \frac{y^2}{8} - \lambda_2$$

Integrating this relation taking into account the first two boundary conditions of (2.4), we find

$$\psi_2 = \frac{y^7}{1120} - \frac{y^6}{480} + \frac{y^5}{480} - \frac{\lambda_2}{6}y^3 + \frac{C_1}{2}y^2$$

Acting similarly, from (2.3) we obtain

$$\theta_2'' = \Pr\left(-\frac{\Pr}{64}y^6 + \frac{\Pr}{32}y^5 - \frac{\Pr}{48}y^4 + \frac{y^6}{160} - \frac{y^5}{80} + \frac{y^4}{96} - \frac{\lambda_2}{2}y^2 + C_1 y\right)$$

whence, after integration and using the third boundary condition of (2.4), it follows that

$$\theta_2 = \Pr\left(-\frac{\Pr}{3584}y^8 + \frac{\Pr}{1344}y^7 - \frac{\Pr}{1440}y^6 + \frac{y^8}{8960} - \frac{y^7}{3360} + \frac{y^6}{2880} - \frac{\lambda_2}{24}y^4 + \frac{C_1}{6}y^3 + A_1 y\right)$$

To find the unknown coefficients C_1 and A_1 and the eigenvalue λ_2, we have three boundary conditions (2.5) which give a linear system, the solution of which is

$$A_1 = \frac{\Pr^2}{840}, \quad C_1 = \frac{\Pr}{48} + \frac{19}{3360}, \quad \lambda_2 = \frac{\Pr}{16} + \frac{5}{224}$$

Hence, the functions ψ_2, θ_2, f_2 of the second approximation take the form

$$\psi_2 = \frac{y^7}{1120} - \frac{y^6}{480} + \frac{y^5}{480} = \left(\Pr + \frac{5}{14}\right)\frac{y^3}{96} + \left(\Pr + \frac{19}{70}\right)\frac{y^2}{96}$$

$$\theta_2 = \Pr\left[\left(\frac{\Pr}{2} - \frac{1}{5}\right)\left(\frac{1}{3} - \frac{y}{8}\right)\frac{y^2}{224} - \left(\Pr - \frac{1}{2}\right)\frac{y^6}{1440} - \left(\Pr + \frac{5}{14}\right)\frac{y^4}{384} + \right.$$
$$\left. + \left(\Pr + \frac{19}{70}\right)\frac{y^3}{288}\right] + \frac{\Pr^2}{840}y$$

$$f_2 = \frac{y^4}{16}(y-1)^2 + \left(\frac{y^2}{10} - \frac{y}{5} + \frac{1}{6}\right)\frac{y^4}{8} - \left(\Pr + \frac{5}{14}\right)\frac{y^2}{16} + \left(\Pr + \frac{19}{70}\right)\frac{y}{24}$$

For the eigenvalue, we obtain

$$\lambda = -\frac{3}{2}m_H + \left(\Pr + \frac{5}{14}\right)\frac{m_H^2}{16} + o(m_H^2)$$

It is clear that, in the second approximation with respect to the Marangoni number but unlike the first approximation, the flow and temperature fields in the channel as well as the pressure coefficient λ now depend on the Prandtl number, that is, the flow field and the heat transfer process in the layer become interdependent.

3. NUMERICAL INTEGRATION

An algorithm for the numerical analysis of system (1.2)–(1.4) for any values of the Marangoni number m_H is proposed below. Reduction of the above-mentioned boundary-value problem to a Cauchy problem (a problem with initial data) is the principal method of investigation.

We shall restrict the treatment to the limiting case of an ideal heat-conducting liquid: $\mathrm{Pr} = 0$. In the limit being considered, the thermal and hydrodynamic aspects of the problem are "uncoupled". In fact, it follows from Eq. (1.3) that $\theta'' = 0$, from which, after satisfying the "thermal" boundary conditions in (1.4), we find that $\theta = 1$, so that the flow of the liquid has no effect on the temperature distribution in it and the temperature profile maintained at the bottom propagates without distortions across the layer.

To determine the flow field, we obtain the boundary-value problem

$$\begin{aligned} &\psi''' + \psi\psi'' = \psi'^2 + \lambda = 0 \\ &y = 0: \quad \psi = 0, \quad \psi' = 0 \\ &y = 1: \quad \psi = 0, \quad \psi'' = m_H \end{aligned} \tag{3.1}$$

The differential equation in (3.1) is subject to the action of a linear group of transformations $\psi(y) \to \bar\psi(\bar y)$ [3]

$$\psi = B^{\gamma_1}\bar\psi, \quad y = B^{\gamma_2}\bar y \tag{3.2}$$

where $B \neq 0$ is a transformation parameter, and γ_1 and γ_2 are exponents which are as yet unknown. The transformed equation has the form

$$\bar\psi''' + B^{\gamma_1+\gamma_2}\bar\psi\bar\psi'' - B^{\gamma_1+\gamma_2}\bar\psi'^2 + B^{-\gamma_1+3\gamma_2}\lambda = 0$$

By requiring it to be invariant under the transformation parameter B, we find that $\gamma_1 = -\gamma_2 = \gamma$ so that

$$\bar\psi''' + \bar\psi\bar\psi'' - \bar\psi'^2 + \bar\lambda = 0$$

where $\bar\lambda = B^{-4\gamma}\lambda$ is a modified eigenvalue.

The boundary conditions at the first point do not change as a result of transformation (3.2): $\bar y = 0$, $\bar\psi = 0$, $\bar\psi' = 0$. The condition on the second derivative $\bar\psi''$ is obviously an insufficient initial condition when $\bar y = 0$. To obtain this condition, we require that the equality $\psi''(0) = B$ must be satisfied (the insufficient initial value before the transformation is equal to the transformation parameter B). After (3.2) has been carried out, this gives $B^{3\gamma}\bar\psi''(0) = B$. On requiring invariance with respect to B in this case also, we find that $\gamma = 1/3$, as a result of which the required initial condition takes the form $\bar\psi''(0) = 1$.

The form of the boundary conditions (3.1) at the terminal policy $y = 1$, after the transformation (3.2), is

$$\bar\psi(B^{1/3}) = 0, \quad \bar\psi''(B^{1/3}) = m_H / B \tag{3.3}$$

The actions which have been described enable us to present the following scheme for constructing the spectrum $\lambda = \lambda(m_H, 0)$ of the boundary-value problem (3.1). The value $\bar\lambda \in (-\infty, +\infty)$ of the modified pressure coefficient is specified and the Cauchy problem

$$\begin{aligned} &\bar\psi''' + \bar\psi\bar\psi'' - \bar\psi'^2 + \bar\lambda = 0 \\ &\bar\psi(0) = 0, \quad \psi'(0) = 0, \quad \bar\psi''(0) = 1 \end{aligned} \tag{3.4}$$

is integrated until the condition $\bar\psi = 0$ is met. Suppose that this occurs at a certain value of the argument $\bar y = \bar y_0$. By the first condition of (3.3), we obtain from this the value of the transformation parameter: $\bar y_0 = B^{1/3}$, $B = \bar y_0^3$. Next, using the second derivative of the solution of problem (3.4) at this point, we find the value of the Marangoni number $m_H = B\bar\psi''(\bar y_0) = \bar\psi''(\bar y_0)\bar y_0^3$. Finally, we obtain the eigenvalue of the initial problem (3.1) in the form $\lambda = B^{4\gamma}\bar\lambda = \bar\lambda\bar y_0^4$.

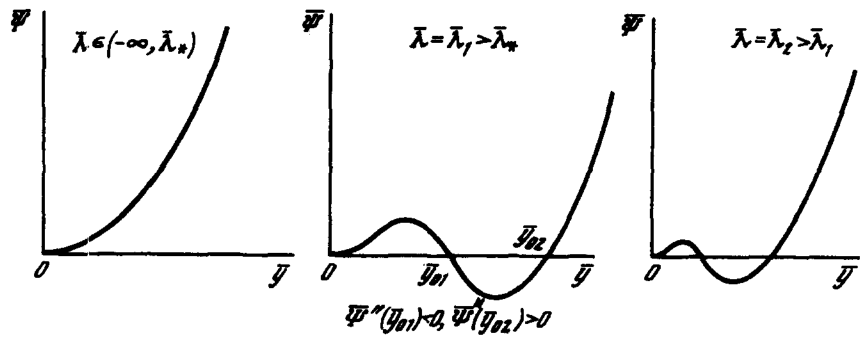

Fig. 1.

Note that the possibility that the root $\bar{y}_0$ of the equation $\bar{\psi}(\bar{y}) = 0$ may be non-unique has been taken into account in the implementation of the algorithm. Moreover, since the transformation parameter B can be of either sign, the initial problem (3.4) must be integrated separately in the domains $\bar{y} \in [0, +\infty)$ and $\bar{y} \in (-\infty, 0]$.

The behaviour of the function $\bar{\psi}(\bar{y})$ when $\bar{y} \in [0, +\infty)$ is shown schematically in Fig. 1 as a function of the modified pressure coefficient $\bar{\lambda}$. It is clear that, in the interval $\bar{\lambda} \in (\bar{\lambda}_*, +\infty)$ ($\bar{\lambda}_* \in (0.75667; 0.75668)$), there are two roots $\bar{y}_{0i}$ $(i = 1, 2)$ of the equation $\bar{\psi}(\bar{y}) = 0$ (apart from the initial value $\bar{y} = 0$) with different signs of $\psi''(\bar{y}_{0i})$ which generate the corresponding branches of the spectrum $\lambda = \lambda(m_H, 0)$ for negative ($\bar{y}_{01}$) and positive ($\bar{y}_{02}$) Marangoni numbers. The function $\bar{\psi}(\bar{y})$ has no zeros in the interval $\bar{\lambda} \in (-\infty, \bar{\lambda}_*)$ when $\bar{y} \in (0, +\infty)$. In the limiting case when $\bar{\lambda} \to +\infty$, it can be approximated in the right-hand neighbourhood of the point $\bar{y} = 0$ by the cubic curve $\varphi(\bar{y}) = -\lambda y^3/6 + \bar{y}^2/2$ which gives $\bar{y}_{01} \sim 3/\bar{\lambda}$, $\lambda \sim 81/\bar{\lambda}^3$, $m_H \sim -54/\bar{\lambda}^3$, that is, $\lambda_{m_H \to 0}$ $-3/2m_H$ and corresponds to the asymptotic form (1.5) found in [2] when $m_H < 0$ (branch 1 in Fig. 3).

When $\bar{y} \in (-\infty, 0)$, the equation $\bar{\psi}(\bar{y}) = 0$ has a unique root $\bar{y}_0$ for any value of the modified pressure coefficient $\lambda \in (-\infty, +\infty)$, where $\bar{\psi}''(\bar{y}_0) < 0$, $m_H > 0$. The behaviour of the function $\bar{\psi}(\bar{y})$ in this case is shown schematically in Fig. 2. Values of $\bar{\lambda} \geqslant 0$ together with the root $\bar{y}_{02}$ (see above) give branch 2 of the curve $\lambda(m_H, 0)$ which corresponds to positive Marangoni numbers. Values of $\bar{\lambda} < 0$ generate branch 3 of the spectrum when $m_H > 0$, one of the ends of which corresponds to the asymptotic form (1.5) while the other, when $\bar{\lambda} \to 0 - 0$, tends to the asymptotic form common with branch 2 when $\lg m_H \cong 3.0582$.

The spectrum $\lambda(m_H, 0)$ which was constructed numerically is shown schematically in Fig. 3 in logarithmic coordinates. The same dependence in (m_H, λ) coordinates is presented in the upper part of Fig. 3. Comparison of the numerical results obtained and the results of asymptotic theory [2], when $m_H \to 0$, $\lambda \to 0$ in the case of the function $\psi'(y) = \bar{y}_0^2\bar{\psi}'(\bar{y})$ which characterizes the longitudinal component u of the velocity of the liquid in the layer when $m_H \cong -5.4046 \times 10^{-11}$, $\lambda = 8.1061 \times 10^{-11}$ ($\bar{\lambda} = 10^4$, branch 1), shows that the relative deviation of the calculated profile from the asymptotic profile for the chosen value of the Marangoni number does not exceed 0.1%. The component u is distinguished from $\psi'(y)$ by the factor xv/H so that, in vertical cross-sections of the layer, which are equidistant from the plane $x = 0$, which is the stream surface, the profiles of the velocity u have mirror symmetry.

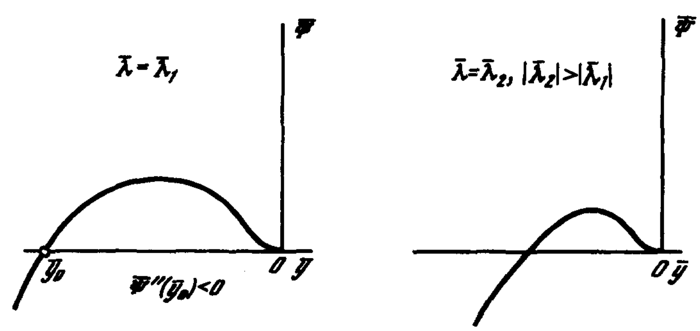

Fig. 2.

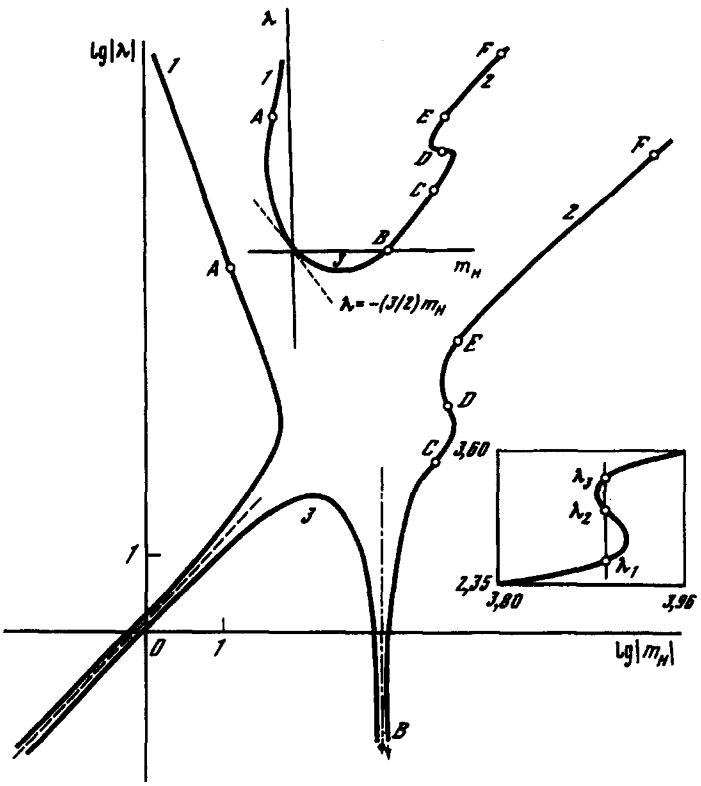

Fig. 3.

One of the characteristic features of non-linear problems of the type considered is the non-uniqueness of the solution, the possibility of which has been discussed in [2] by the analogy between boundary-value problem (1.2)–(1.4) and the problem investigated numerically in [4], where the flow of a liquid in plane and cylindrical channels with walls which experience a linear extension with constant velocity was considered. It was established in [4] that from one to three values of the pressure coefficient may correspond to a fixed value of the Reynolds number and, in the axially symmetric case, there may be no solution at all over a certain range of Reynolds numbers.

It has been established above that the spectrum $\lambda\,(m_H, 0)$ of problem (3.1) possesses similar properties. In particular, along branch 1 ($m_H < 0$) there are at most two solutions over the range of modified eigenvalues which has been considered $\overline{\lambda} \in (0.75668; 10^4)$ ($\lambda \cong 6.9111 \times 10^{11}, m_H = -0.13553$ correspond to the first value close to $\overline{\lambda}_*$ and $\lambda \cong 8.1061 \times 10^{-11}$, $m_H = -5.4046 \times 10^{-11}$ to the second value, so that the representative point actually lies on the asymptote (1.5), that is, the dashed line in Fig. 3). In the case of negative values of the Marangoni number with a sufficiently high modulus ($\lg|m_H| \gg 1.685$), the boundary-value problem (3.1) does not have solutions for any values of λ whatsoever, which physically means that it is impossible to achieve steady flow conditions.

Along branches 2 and 3 ($m_H > 0$) there are from one to three values of the eigenvalue λ which correspond to a fixed value of the Marangoni number (there is one solution on branch 3). A segment of branch 2 with a non-unique solution is shown on an enlarged scale in Fig. 3. We also note that there are at least two solutions with a zero eigenvalue: one corresponds to the asymptotic form (1.5), m_H = 0, while the second corresponds to the Marangoni number $m_H \cong 1143.3$ (the common vertical asymptote of branches 2 and 3).

4. THE FLOW PATTERN

Data on the nature of the circulation of the liquid which occurs in the layer as a result of the action of thermocapillary forces are shown in Fig. 4. The profiles of the horizontal component of the velocity are shown for a number of distinctive points of the spectrum $\lambda\,(m_H, 0)$ (the values of the function

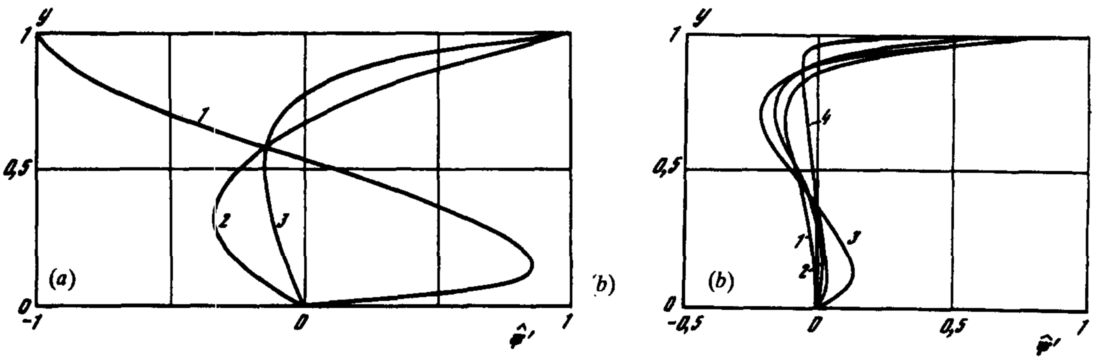

Fig. 4.

$\hat{\psi}'(y) = \psi'(y)/(\max_{y \in [0,1]} \psi'(y))$ are plotted along the abscissa). Curve 1 in Fig. 4(a) corresponds to point A on branch 1 ($\bar{\lambda} = 0.78$; $m_H < 0$). A reverse flow in the layer occurs approximately in the middle of its depth and the maximum velocities in opposite directions are close in magnitude. Velocity profiles, corresponding to two solutions in the case of a zero eigenvalue: the limiting profile from [2] when $\lambda \to 0$, $m_H \to 0$ (curve 2) and the profile corresponding to the limiting position of point B in Fig. 3: $\lambda = 0$, $m_H = 1143.3$ (curve 3) are also compared in Fig. 4(a). The dynamics of the change in the flow pattern accompanying motion along branch 2 ($m_H > 0$) is represented by curves 1–4 in Fig. 4(b) which correspond to the points C, D, E and F ($\lambda^- = 10^5$; 1.2; 0.9; 0.76). The most interesting feature is the emergence of a "three-layer sandwich" structure in the flow as m_H increases and the tendency for the liquid near the free surface to be accelerated by the surface tension forces relative to its more slowly moving internal layers when $m_H \to +\infty$.

In concluding, we note that, by using the procedure of continuation with respect to a parameter [3], with the Prandtl number appearing as this parameter in the problem being considered, it is possible to trace the evolution of the spectrum λ (m_H, 0) for non-zero values of Pr when the thermal and hydrodynamic fields interact in a more complex manner according to the complete system (1.2)–(1.4).

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